

Symbolic approach and induction in the Heisenberg group

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Abstract. We associate a homomorphism in the Heisenberg group to each hyperbolic unimodular automorphism of the free group on two generators. We show that the first return-time of some flows in "good" sections, are conjugate to niltranslations, which have the property of being self-induced.

We introduce an extension of the well known connection between geometric systems and symbolic systems (broken lines, Rauzy fractals, etc), to the non commutative setting, i.e. Heisenberg group. The symbolic objects considered are automorphisms of the free group on m generators \mathbb{F}_m . If G is an arbitrary group generated by m generators, then there exists a surjective group homomorphism π mapping \mathbb{F}_m onto G . The goal is then to translate the action of an automorphism σ on \mathbb{F}_m , in an application on the group G , through the application π .

Let σ be a **substitution** on m letters, i.e. a positive endomorphisms of the free group on m generators \mathbb{F}_m . Suppose there exists a unique infinite word $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ such that $\sigma(\mathbf{u}) = \mathbf{u}$. A natural construction exists, which associates to the substitution σ , a sequence $(x_n)_n$ of elements of G , such that the n th term of the sequence, is the projection by π of the prefix $u_0 \dots u_n$. (i.e. $x_n = \pi(u_0 \dots u_n)$). This sequence of elements is called the **broken line** in G associated to the substitution σ . When G is the group \mathbb{Z}^m , the abelianisation of the free group on m generators, under some assumptions on the substitution, the closure of a projection of the broken line is a compact set of \mathbb{R}^{m-1} , called a **Rauzy fractal**. We can then measurably conjugate the symbolic dynamical system generated by \mathbf{u} , with an exchange of pieces of this fractal. There are many generalizations of this construction, especially for free groups with more generators (see [3], [5] and [7]).

Another method is to translate the action of the substitution to the group G , in a way which is consistent with the morphism π . A topological group G with m generators will be called **adapted for automorphisms of the free group** \mathbb{F}_m endowed with its natural topology, if the map π is continuous and if there is a continuous and surjective homomorphism $\mathfrak{S} : \text{Aut}(\mathbb{F}_m, \mathbb{F}_m) \rightarrow \text{Aut}(G, G)$ such that for any $\sigma \in \text{Hom}(\mathbb{F}_m, \mathbb{F}_m)$, the following diagram commutes :

$$\begin{array}{ccc} \mathbb{F}_m & \xrightarrow{\sigma} & \mathbb{F}_m \\ \pi \downarrow & & \downarrow \pi \\ G & \xrightarrow{\mathfrak{S}_\sigma} & G \end{array}$$

We shall then say that \mathfrak{S}_σ is the **factorization** of σ . Since π is surjective, it is possible that the same morphism is associated with two different automorphisms. It is important to note right now, that with the point of view that we adopt in this work, two automorphisms that can be factored in the same way on G , will be indistinguishable. Let u and v be two elements of the free group \mathbb{F}_m . We denote by $[u, v] = uvu^{-1}v^{-1}$ the **commutator** of u and v . The endomorphisms σ of free groups satisfy the relation:

$$\sigma([u, v]) = [\sigma(u), \sigma(v)].$$

This relation gives hope to obtain interesting results considering nilpotent groups, defined from relations of commutators in the free group.

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We consider the situation where G is the discret Heisenberg group and will give some results for the group \mathbb{F}_2 . We begin by recalling some results related to the Heisenberg group in Section 1 and we introduce transformations of this space such as the **nilflows** and the **niltranslations**.

Proposition 1. *The set of matrices with integer coefficients form a lattice of the Heisenberg group which is adapted for automorphisms of the free group \mathbb{F}_2 .*

There is no object known at this time, which corresponds to the Rauzy fractal. But we believe such an object exists. We obtain some results in this direction in Section 3, where we study a family of niltranslations connected with the "Fibonacci substitution". The fact that these niltranslations come from substitutions, yields self-induced dynamical systems. The self-induction property corresponds to the self-similarity of the Rauzy fractal under renormalization. We show:

Proposition 2. *Let ϕ be the golden mean. The dynamical system given by the application:*

$$\begin{aligned} (\mathbb{R}/\mathbb{Z})^2 &\longrightarrow (\mathbb{R}/\mathbb{Z})^2 \\ (y, z) &\mapsto \left(y + \frac{1}{\phi^2}, z + y - \frac{1}{2\phi^3} \right) \end{aligned} \text{ is self-induced, minimal and uniquely ergodic.}$$

The Heisenberg group has in its automorphisms group, some semi-simple and hyperbolic elements, that stabilize discrete Heisenberg group Γ , and preserve the center. Let G , be the group of unimodular automorphisms of the Heisenberg space. The space $G/\text{stab}_G(\Gamma)$ is then a natural lattice space. The set of one-parameter flows is identified with the Lie algebra of the group, and we can then consider it, as a flat bundle over the moduli space. The flow generated by a one-parameter group of semi-simple hyperbolic elements on G , induced on this bundle, a flow called **the renormalization flow**. L. Flaminio and G. Forni study this flow in [8]. They deduce results on the distribution of flows in Heisenberg space by considering a co-homological equation.

We are interested by the periodic orbits of the renormalization flow and give an explicit calculation of the renormalized flow. In Proposition 1, we show that the periodic points of the renormalization flow arise naturally from automorphisms acting on \mathbb{F}_2 .

We construct sections of flows adapted to these automorphisms. We will see that the first return of these flows into these sections, have remarkable properties. The existence of such sections is not obvious, and we are currently unable to generalize this constructions to higher dimensions. These applications are conjugate to niltranslations. We obtain the following result:

Theorem 1. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a matrix with integer coefficients such that $|\det(M)| = 1$.*

Assume that this matrix admits an eigenvalue λ with modulus $|\lambda| > 1$. We denote by (α, β) the eigenvector associated to λ such that $\alpha + \beta = 1$. For every pair of integers (n, m) , let:

$$\gamma = \frac{\alpha}{\lambda - \det(M)} \left(n - \frac{AC}{2} \right) + \frac{\beta}{\lambda - \det(M)} \left(m - \frac{BD}{2} \right). \quad (1)$$

Then, with the notations which we will introduce, the niltranslation by the element $\begin{bmatrix} \alpha \\ \beta \\ \gamma + \frac{\alpha\beta}{2} \end{bmatrix}$ on the nilmanifold

$$\left\{ \begin{bmatrix} x \\ -x + n \\ z \end{bmatrix}; \begin{matrix} (x, z) \in \mathbb{R}^2 \\ n \in \mathbb{Z} \end{matrix} \right\} / \left\{ \begin{bmatrix} n \\ m \\ p \end{bmatrix}; (n, m, p) \in \mathbb{Z}^3 \right\} \text{ is self-induced, minimal and uniquely ergodic.}$$

In particular, this theorem states that each niltranslation which is the first return map (with constant return time 1) of a nilflow periodic under renormalization, is self-induced. Then, a naturel question arise : *Do the self-induced niltranslation come from a periodic nilflow under renormalization ?* We will see that the answer to this question is no and we start Section 5 by exhibiting a counterexample. We also raise another difficulty. We will see that it is possible that the niltranslations can be self-induced in areas that do not project well on the abelianisation.

1 The Heisenberg group

We recall here some properties of the **Heisenberg group** $\mathbb{H}_3(\mathbb{R})$, denoted \mathbf{X} , of real upper triangular 3×3 matrices, with “1s” on the diagonal. The group law is given by:

$$\mathbf{x} \bullet \mathbf{x}' = \begin{bmatrix} x+x' \\ y+y' \\ z+z'+xy' \end{bmatrix} \text{ where } \mathbf{x} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \text{ We get } \mathbf{x}^{-1} = \begin{bmatrix} -x \\ -y \\ xy-z \end{bmatrix}.$$

The identity element of the group is the identity matrix, denoted by $\mathbf{1}$. The **commutator** of elements \mathbf{x} and \mathbf{y} is: $[\mathbf{x}, \mathbf{y}] = \mathbf{x} \bullet \mathbf{y} \bullet \mathbf{x}^{-1} \bullet \mathbf{y}^{-1}$. The **center** of the group is:

$$\mathbf{Z} = \{\mathbf{x} \in \mathbf{X}; \forall \mathbf{y} \in \mathbf{X}, [\mathbf{x}, \mathbf{y}] = \mathbf{1}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbf{X}; z \in \mathbb{R} \right\}.$$

We denote by $\mathbf{p} : \mathbf{X} \rightarrow \mathbb{R}^2$, the group homomorphism defined by: $\mathbf{p}(\mathbf{x}) = (x, y)$. The following sequence is exact:

$$1 \longrightarrow \mathbf{Z} \xrightarrow{i} \mathbf{X} \xrightarrow{\mathbf{p}} \mathbb{R}^2 \longrightarrow 1.$$

We endow the space \mathbf{X} , with a metric \mathbf{d} , which is invariant by left multiplication. (i.e. $\forall (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbf{X}^3$: $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \mathbf{d}(\mathbf{z} \bullet \mathbf{x}, \mathbf{z} \bullet \mathbf{y})$). It will be defined from the **group norm**:

$$\|\cdot\|_{\mathbf{X}} : \mathbf{X} \longrightarrow \mathbb{R}^+ \text{ defined by } \left\| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\|_{\mathbf{X}} = \left((x^2 + y^2)^2 + \left(z - \frac{xy}{2} \right)^2 \right)^{\frac{1}{4}}.$$

The application $\|\cdot\|_{\mathbf{X}}$ is a group norm because it verifies the following three properties: $\|\mathbf{x}\|_{\mathbf{X}} = 0$ if and only if $\mathbf{x} = \mathbf{1}$, $\|\mathbf{x}\|_{\mathbf{X}} = \|\mathbf{x}^{-1}\|_{\mathbf{X}}$ and $\|\mathbf{x} \bullet \mathbf{y}\|_{\mathbf{X}} \leq \|\mathbf{x}\|_{\mathbf{X}} + \|\mathbf{y}\|_{\mathbf{X}}$ for all $(\mathbf{x}, \mathbf{y}) \in \mathbf{X}^2$. The metric \mathbf{d} is defined for all $(\mathbf{x}, \mathbf{y}) \in \mathbf{X}^2$ by: $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}^{-1} \bullet \mathbf{y}\|_{\mathbf{X}}$.

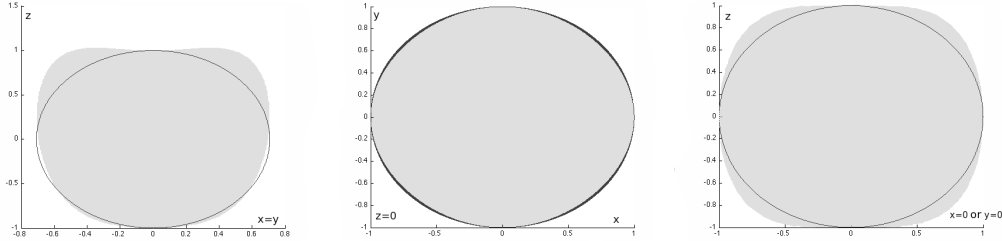


Figure 1: We represent the unit ball of (\mathbf{X}, \mathbf{d}) and the unit sphere of the standard Euclidean space \mathbb{R}^3 in subspaces of \mathbf{X} consisting of matrices $[x, y, z]$ satisfying respectively: $x = y$, $z = 0$, and $x = 0$.

Although the metric \mathbf{d} and the standard Euclidean metric are different, they induce the same topology on \mathbb{R}^3 .

Since $\mathbf{Z} = [\mathbf{X}, \mathbf{X}]$, the space $(\mathbf{X}, \bullet, \mathbf{d})$ is a **nilpotent Lie group** of rank 2. It can be endowed with a differentiable structure. The tangent space to \mathbf{X} in $\mathbf{1}$, which is by definition its **Lie Algebra**, is:

$$\mathfrak{g} = \left\{ \mathfrak{x} = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}; (\alpha, \beta, \gamma) \in \mathbb{R}^3 \right\}. \text{ The elements of } \mathfrak{g} \text{ will be denoted } \mathfrak{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Since the space (\mathbf{X}, \mathbf{d}) is connected, the exponential is a diffeomorphism from this space to Lie algebra \mathfrak{g} .

$$\exp : \mathfrak{g} \longrightarrow \mathbf{X} \quad \mathfrak{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \mapsto \mathbf{1} + \mathfrak{x} + \frac{1}{2}\mathfrak{x}^2 = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \log : \mathbf{X} \longrightarrow \mathfrak{g} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{pmatrix} x \\ y \\ z - \frac{xy}{2} \end{pmatrix}.$$

The **Lie bracket** in the Lie algebra is defined by: $[\mathbf{x}, \mathbf{x}'] = \frac{1}{2} \log([\exp \mathbf{x}, \exp \mathbf{x}'])$.

With these notations, $\exp(\mathbf{x} + \mathbf{x}') \bullet \exp([\mathbf{x}, \mathbf{x}']) = \exp \mathbf{x} \bullet \exp \mathbf{x}'$ and $\log(\mathbf{x} \bullet \mathbf{x}') = \log \mathbf{x} + \log \mathbf{x}' + \log([\mathbf{x}, \mathbf{x}'])$.

For any element $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ of the Lie algebra \mathfrak{g} , we denote by:

$$\mathbf{G}_{\mathbf{x}} = \{\exp(t \cdot \mathbf{x}) ; t \in \mathbb{R}\} = \left\{ \begin{bmatrix} \alpha t \\ \beta t \\ \gamma t + \frac{\alpha\beta}{2}t^2 \end{bmatrix} ; t \in \mathbb{R} \right\} \text{ with the convention } t \cdot \mathbf{x} = \begin{pmatrix} \alpha t \\ \beta t \\ \gamma t \end{pmatrix}. \quad (2)$$

These are the **1 parameter sub-groups** of \mathbf{X} .

Let $\mathbb{H}_3(\mathbb{Z}) = \mathbf{\Gamma}$ be the sub-group of \mathbf{X} consisting of matrices with integer coefficients. The following sequence is exact:

$$1 \longrightarrow \mathbf{Z} \cap \mathbf{\Gamma} \xrightarrow{i} \mathbf{\Gamma} \xrightarrow{p} \mathbb{Z}^2 \longrightarrow 1.$$

The metric \mathbf{d} induces a metric $\underline{\mathbf{d}}$ on the quotient space $\mathbf{X} \setminus \mathbf{\Gamma}$ denoted $\underline{\mathbf{X}}$. The space \mathbf{X} acts isometrically by left translation on $\underline{\mathbf{X}}$. There is a unique probability measure invariant by this action, called the **Haar measure**. By definition, $(\underline{\mathbf{X}}, \underline{\mathbf{d}})$ is a **nilmanifold**. The space $\underline{\mathbf{X}}$ is topologically isomorphic to the space $[0; 1]^3$, with the following identifications:

$$\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \sim \begin{bmatrix} 1 \\ y \\ z \end{bmatrix}, \quad \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \sim \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \sim \begin{bmatrix} x \\ 1 \\ z \end{bmatrix} \text{ and } \begin{bmatrix} x \\ 1 \\ z \end{bmatrix} \sim \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}.$$

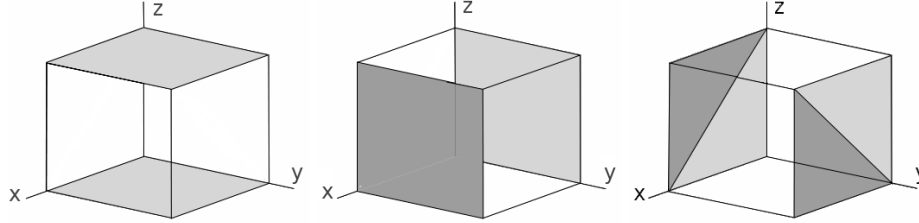


Figure 2: Identification of the faces of the standard unit cube to get the nilmanifold $\underline{\mathbf{X}}$.

The Haar measure of the space $\underline{\mathbf{X}}$, immersed in this fundamental domain, is the standard Lebesgue measure, denoted λ^3 . There are three types of dynamical systems acting on spaces \mathbf{X} and $\underline{\mathbf{X}}$ natural to consider and preserving the measure.

The first family of applications is composed of continuous homomorphisms of \mathbf{X} . Since the work of G. Gelbrich in [11], we know that they are of the following form:

$$\mathcal{L} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} ax + by \\ cx + dy \\ \frac{ac}{2}x^2 + (e - \frac{ac}{2})x + \frac{bd}{2}y^2 + (f - \frac{bd}{2})y + bcxy + (ad - bc)z \end{bmatrix}. \quad (3)$$

\mathcal{L} preserve the lattice $\mathbf{\Gamma}$ when the coefficients (a, b, c, d, e, f) are integers. In this case, the application \mathcal{L} also acts on the quotient space $\underline{\mathbf{X}}$. In addition, these applications preserve the Haar measure, if the coefficients satisfy the equation: $|ad - bc| = 1$.

We are also interested in the action of 1 parameter subgroups on the space \mathbf{X} , given by equation (2):

$$\Phi_{\mathbf{x}}^t : \mathbf{x} \mapsto \mathbf{g}_{\mathbf{x}}^t \bullet \mathbf{x}, \text{ where } \mathbf{g}_{\mathbf{x}}^t = \exp(t \cdot \mathbf{x}) = \begin{bmatrix} \alpha t \\ \beta t \\ \gamma t + \frac{\alpha\beta}{2}t^2 \end{bmatrix} \text{ with } t \in \mathbb{R} \text{ and } \mathbf{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathfrak{g}, \quad (4)$$

and their discrete time analogue, the action by left translations:

$$T_{\mathbf{r}} : \begin{array}{ccc} \mathbf{X} & \rightarrow & \mathbf{X} \\ \mathbf{x} & \mapsto & \mathbf{g}_{\mathbf{r}} \bullet \mathbf{x} \end{array} \text{ with } \mathbf{r} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathfrak{g} \text{ and } \mathbf{g}_{\mathbf{r}} = \exp(\mathbf{r}) = \begin{bmatrix} \alpha \\ \beta \\ \gamma + \frac{\alpha\beta}{2} \end{bmatrix}. \quad (5)$$

We will also denote these maps $\Phi_{\alpha,\beta,\gamma}$ and $T_{\alpha,\beta,\gamma}$. These applications act naturally on the quotient space $\underline{\mathbf{X}}$, denoted them by \underline{T} and $\underline{\Phi}$. These classes of systems, called **niltranslations** and **nilflows**, have been widely studied. Let us quote here two central results. For $\underline{\mathbf{x}} \in \underline{\mathbf{X}}$, we put $\mathcal{O}(\underline{\mathbf{x}}) = \{\underline{T}^n(\underline{\mathbf{x}}); n \in \mathbb{N}\}$.

Theorem A (E. Lesigne [15]). *The system $(\mathcal{O}(\underline{\mathbf{x}}), \underline{T})$ is minimal and uniquely ergodic.*

Theorem B (L. Auslander, L. Green and F. Hahn [13], [6]). *The flow $(\Phi_{\alpha,\beta,\gamma}^t)_t$ on $\underline{\mathbf{X}}$ is minimal if and only if it is uniquely ergodic if and only if the coefficients α and β are linearly independent.*

We consider the flow Ψ^t defined by: $\mathbf{c}^t = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$ and $\Psi^t(\mathbf{x}) = \mathbf{c}^t \bullet \mathbf{x}$ for $t \in \mathbb{R}$.

Lemma 1. $\Phi_{\alpha',\beta',\gamma'}^s \circ \Phi_{\alpha,\beta,\gamma}^t = \Phi_{\alpha,\beta,\gamma}^t \circ \Phi_{\alpha',\beta',\gamma'}^s \circ \Psi^{\Delta(t-s)}$ where $\Delta = \beta\alpha' - \beta'\alpha$. In particular, $\Psi^s \circ \Phi_{\alpha,\beta,\gamma}^t = \Phi_{\alpha,\beta,\gamma}^t \circ \Psi^s$.

Proof. Just calculate the following expressions:

$$\begin{aligned} \Phi_{\alpha',\beta',\gamma'}^s \circ \Phi_{\alpha,\beta,\gamma}^t \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x + t\alpha + s\alpha' \\ y + t\beta + s\beta' \\ z + yt\alpha + (y + t\beta)s\alpha' + \gamma t + \gamma' s + \frac{\alpha\beta}{2}t^2 + \frac{\alpha'\beta'}{2}s^2 \end{bmatrix} \\ \text{and } \Phi_{\alpha,\beta,\gamma}^t \circ \Phi_{\alpha',\beta',\gamma'}^s \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x + s\alpha' + t\alpha \\ y + s\beta' + t\beta \\ z + ys\alpha' + (y + s\beta')t\alpha + \gamma' s + \gamma t + \frac{\alpha'\beta'}{2}s^2 + \frac{\alpha\beta}{2}t^2 \end{bmatrix}. \end{aligned}$$

□

The group norm verifies some properties with respect to the introduced objects. For any element $\mathbf{x} \in \mathbf{X}$, the flow $\Phi_{\log \mathbf{x}}$ is the unique flow satisfying $\Phi^1(\mathbf{0}) = \mathbf{x}$. If $\mathbf{x} = [x, y, z]$, the group norm verifies:

$$\|\Phi_{\log \mathbf{x}}^t(\mathbf{1})\|_{\mathbf{X}} = \left(t^4 (x^2 + y^2)^2 + t^2 \left(z - \frac{xy}{2} \right)^2 \right)^{\frac{1}{4}}.$$

For every real t , we also consider the expansion of space $\mathcal{D}^t : \mathbf{X} \rightarrow \mathbf{X}$, such that $\mathcal{D}^t([x, y, z]) = [xt, yt, zt^2]$. The group norm verifies: $\|\mathcal{D}^t \mathbf{x}\|_{\mathbf{X}} = |t| \cdot \|\mathbf{x}\|_{\mathbf{X}}$ for $\mathbf{x} \in \mathbf{X}$.

A significant difference with the abelian situation, is that for every real $t \notin \{0, 1\}$, the application of \mathbf{X} into itself defined by: $\mathbf{x} \mapsto \Phi_{\log \mathbf{x}}^t(\mathbf{1})$, is not a group homomorphism of \mathbf{X} . For more details on the left invariant metric of this group, we refer to [2], [12], [14] and [16].

2 Symbolic approach

We start by proving Proposition 1, which makes the link between the automorphisms of the free group on two generators and the morphisms of the lattice $\mathbb{H}_3(\mathbb{Z}) = \Gamma$. The generators of this lattice will be noted:

$$\mathbf{n}_a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{n}_b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \text{ Also let } \mathbf{n}_{a^{-1}} = \mathbf{n}_a^{-1}, \mathbf{n}_{b^{-1}} = \mathbf{n}_b^{-1} \text{ and } \mathbf{n} = [\mathbf{n}_a, \mathbf{n}_b] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbf{Z} \cap \Gamma.$$

Let σ be a automorphism of the free group \mathbb{F}_2 . It can be written

$$\sigma : \begin{cases} a & \longrightarrow \xi_1 \dots \xi_{l_a} \\ b & \longrightarrow \zeta_1 \dots \zeta_{l_b} \end{cases} \quad \text{with each } (\xi_i)_{1 \leq i \leq l_a} \text{ and } (\zeta_i)_{1 \leq i \leq l_b} \text{ in } \{a, b, a^{-1}, b^{-1}\}. \quad (6)$$

We associate to σ the endomorphism \mathfrak{S}_σ of Γ defined by $\mathfrak{S}_\sigma(\mathbf{n}_a) = \mathbf{n}_{\xi_1} \dots \mathbf{n}_{\xi_{l_a}}$ and $\mathfrak{S}_\sigma(\mathbf{n}_b) = \mathbf{n}_{\zeta_1} \dots \mathbf{n}_{\zeta_{l_b}}$. (7)

This object is well defined since $\mathfrak{S}_\sigma(\mathbf{n}) = \mathfrak{S}_\sigma(\mathbf{n}_a) \cdot \mathfrak{S}_\sigma(\mathbf{n}_b) \cdot \mathfrak{S}_\sigma(\mathbf{n}_a)^{-1} \cdot \mathfrak{S}_\sigma(\mathbf{n}_b)^{-1}$.

Proof of Proposition 1. The application $\mathfrak{S} : \text{Aut}(\mathbb{F}_2) \rightarrow \text{End}(\Gamma)$ is a morphism. We will show that $\mathfrak{S}(\text{Aut}(\mathbb{F}_2)) = \text{Aut}(\Gamma)$. We start by showing that for every $\sigma \in \text{Aut}(\mathbb{F}_2)$, then $\mathfrak{S}_\sigma \in \text{Aut}(\Gamma)$. We note M the action of $\mathbf{p} \circ \mathfrak{S}_\sigma$ on \mathbb{R}^2 . From Equation (3), we know that $\mathfrak{S}_\sigma(\mathbf{n}) = \det(M)\mathbf{n}$. Since $\det(M) \in \{-1, 1\}$, \mathfrak{S}_σ is an automorphism of Γ . For more details, we refer to [9] and [17]. *It only remains to verify that the map \mathfrak{S} is surjective.*

Consider an endomorphism \mathcal{L} given by equation (3) of Section 1. We have $\mathcal{L}(\mathbf{n}_a) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathcal{L}(\mathbf{n}_b) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

We define the automorphisms $\sigma_1, \sigma_2, \sigma_3$ and σ_4 , defined by:

$$\sigma_1 : \begin{cases} a & \longrightarrow ab \\ b & \longrightarrow b \end{cases}, \sigma_2 : \begin{cases} a & \longrightarrow ab \\ b & \longrightarrow a \end{cases}, \sigma_3 : \begin{cases} a & \longrightarrow a \\ b & \longrightarrow ba \end{cases} \quad \text{and } \sigma_4 : \begin{cases} a & \longrightarrow b \\ b & \longrightarrow ab \end{cases}.$$

We write $\mathfrak{S}_i = \mathfrak{S}_{\sigma_i}$, for $i \in \{1, 2, 3, 4\}$. It is conventional to verify that there exists an integer k , $(n, m) \in \mathbb{Z}^2$, $(u_j) \in \{1, 4\}^k$, and $(\epsilon_j) \in \{-1, 1\}^k$ such that

$$\prod_{j=1}^k \mathfrak{S}_{u_j}^{\epsilon_j} \circ \mathfrak{S}(\mathbf{n}_a) = \mathbf{n}_a \bullet \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix} \quad \text{and} \quad \prod_{j=1}^k \mathfrak{S}_{u_j}^{\epsilon_j} \circ \mathfrak{S}(\mathbf{n}_b) = \mathbf{n}_b \bullet \begin{bmatrix} 0 \\ 0 \\ m \end{bmatrix}.$$

Then define the following automorphisms:

$$\sigma_5 : \begin{cases} a & \longrightarrow b^{-1}ab \\ b & \longrightarrow b \end{cases}, \sigma_6 : \begin{cases} a & \longrightarrow a \\ b & \longrightarrow a^{-1}ba \end{cases} \quad \text{and } \mathfrak{S}_i = \mathfrak{S}_{\sigma_i}, \text{ for } i \in \{5, 6\}.$$

By a calculation, we can verify that

$$(\mathfrak{S}_5)^n \circ (\mathfrak{S}_6)^{-m}(\mathbf{n}_a) = \mathbf{n}_a \bullet \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix} \quad \text{and} \quad (\mathfrak{S}_6)^n \circ (\mathfrak{S}_5)^{-m}(\mathbf{n}_b) = \mathbf{n}_b \bullet \begin{bmatrix} 0 \\ 0 \\ m \end{bmatrix}.$$

So, with $\sigma = \prod_{j=1}^k \sigma_{u_{k-j+1}}^{-\epsilon_{k-j+1}} \circ (\mathfrak{S}_6)^n \circ (\mathfrak{S}_5)^{-m}$, we have $\mathcal{L} = \mathfrak{S}_\sigma$. □

Throughout this work, we deal with the general case. However, we will treat the **Fibonacci substitution** to illustrate our results:

$$\tau : \begin{cases} a & \rightarrow ab, \\ b & \rightarrow a. \end{cases}$$

We denote by $u = (u_k)_{k \geq 1} = abaaaba \dots \in \{a, b\}^\mathbb{N}$ the infinite word, fixed point of this substitution, and $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean. We begin with the Fibonacci substitution. We define a sequence $(\mathbf{x}_k)_{k \geq 0} \in \mathbf{X}^\mathbb{N}$ as follows:

$$\mathbf{x}_0 = \mathbf{1} \text{ and for } k \geq 1: \mathbf{x}_{k+1} = \mathbf{x}_k \bullet \mathbf{n}_{u_k}.$$

We call this sequence, **the broken line** associated with the substitution τ in \mathbf{X} . For any integer k , we write:

$$\mathbf{x}_k = \mathbf{1} \bullet \prod_{i=1}^k \mathbf{n}_{u_i} = \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix}.$$

A direct calculation shows that $a_0 = b_0 = c_0 = 0$, and for any integer $k \geq 1$:

$$a_k = \#\{1 \leq i \leq k ; u_i = a\} , b_k = \#\{1 \leq i \leq k ; u_i = b\} \text{ and } c_k = \#\{1 \leq i < j \leq k ; u_i = a \text{ and } u_j = b\} .$$

For any integer k , the quantity c_k can be viewed “geometrically” by the area of the gray zone in Figure 3.

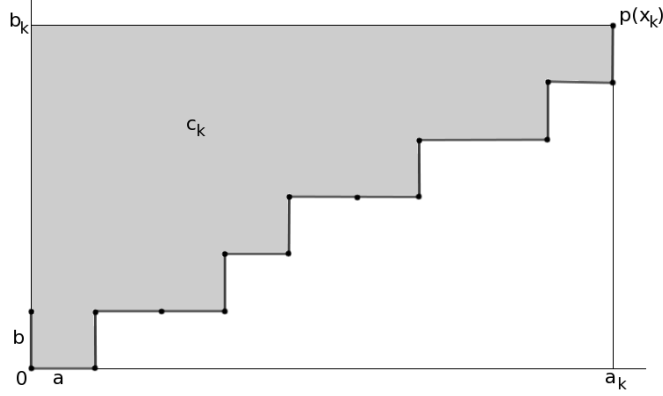


Figure 3: Projections by \mathbf{p} of the broken line $(\mathbf{x}_k)_k$ in \mathbb{R}^2 .

The challenge is to find an element $\mathbf{g} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in \mathbf{X}$ such that the sequence $(\mathbf{g}^k \bullet \mathbf{x}_k)_k$ is bounded.

In particular, in order to bound the sequence of elements $(\mathbf{p}(\mathbf{g}^k(\alpha, \beta, \gamma) \bullet \mathbf{x}_k))_k$ of \mathbb{R}^2 , the element \mathbf{g} should be choosen as:

$$\mathbf{g}_\theta = \begin{pmatrix} 1 & \frac{-1}{\phi} & \theta \\ 0 & 1 & \frac{-1}{\phi^2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} \frac{-1}{\phi} \\ \frac{-1}{\phi^2} \\ \theta \end{bmatrix} .$$

For this reason, we focus in Section 3 on the left action of matrices \mathbf{g}_θ on the quotient space $\underline{\mathbf{X}}$.

For any automorphism σ of \mathbb{F}_2 , we will use the following notation:

$$M_\sigma = [m^{\epsilon, \epsilon'}]_{(\epsilon, \epsilon') \in \{a, b\}^2} = \mathbf{p} \circ \mathfrak{S}_\sigma \text{ and } \mathfrak{S}_\sigma : \begin{matrix} \mathbf{X} & \rightarrow \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} & \mapsto \end{matrix} \begin{matrix} \mathbf{X} \\ \begin{bmatrix} m^{a,a}x + m^{a,b}y \\ m^{b,a}x + m^{b,b}y \\ \det(M_\sigma)z + P_\sigma(x, y) \end{bmatrix} \end{matrix} , \quad (8)$$

$$\text{where } P_\sigma(x, y) = \frac{m^{a,a}m^{b,a}}{2}x(x-1) + \frac{m^{a,b}m^{b,b}}{2}y(y-1) + m^{a,b}m^{b,a}xy + n_a^{a,b}x + n_b^{a,b}y.$$

We notice immediately that the map \mathfrak{S}_σ is invertible if and only if the matrix M_σ is itself invertible. *We will always assume this to hold.* For the Fibonacci substitution, this automorphism is:

$$\mathfrak{S}_\tau : \begin{matrix} \mathbf{X} & \rightarrow \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} & \mapsto \end{matrix} \begin{matrix} \mathbf{X} \\ \begin{bmatrix} x+y \\ x \\ -z + x(x+1)/2 + xy \end{bmatrix} \end{matrix} . \quad (9)$$

In the proof of the following proposition we will see that under some assumptions on the matrix M_σ , we can associate to these automorphisms, some characteristic flows.

Proposition 3. *Let λ be a real eigenvalue of the matrix M_σ which is not equal to the determinant of the matrix M_σ . Let (α, β) be an eigenvector of the matrix associated to the eigenvalue λ . Then, there exists a unique real γ , such that the flow $\Phi_{\alpha, \beta, \gamma}^t$ satisfies:*

$$\mathfrak{S} \circ \Phi_{\alpha, \beta, \gamma}^t \circ \mathfrak{S}^{-1} = \Phi_{\alpha, \beta, \gamma}^{\lambda t}.$$

The value of γ is $\gamma = \frac{\alpha}{\lambda - \det(M_\sigma)} \left(n_a^{a,b} - \frac{m^{a,a} m^{b,a}}{2} \right) + \frac{\beta}{\lambda - \det(M_\sigma)} \left(n_b^{a,b} - \frac{m^{b,a} m^{b,b}}{2} \right).$

Proof. We denote the flow $\Phi_{\alpha, \beta, \gamma}$ defined in (4) by Φ . A direct calculation gives:

$$\mathfrak{S} \circ \Phi^t \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m^{a,a}x + m^{a,b}y + t(m^{a,a}\alpha + m^{a,b}\beta) \\ m^{b,a}x + m^{b,b}y + t(m^{b,a}\alpha + m^{b,b}\beta) \\ \left[z + yt\alpha + \gamma t + \frac{\alpha\beta}{2}t^2 \right] \det(M_\sigma) + P_\sigma(x + t\alpha, y + t\beta) \end{bmatrix}$$

and $\Phi^{\lambda t} \circ \mathfrak{S} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m^{a,a}x + m^{a,b}y + \lambda t\alpha \\ m^{b,a}x + m^{b,b}y + \lambda t\beta \\ z \det(M_\sigma) + \lambda t\alpha(m^{b,a}x + m^{b,b}y) + \gamma\lambda t + \frac{\alpha\beta}{2}(\lambda t)^2 + P_\sigma(x, y) \end{bmatrix}.$

It is therefore necessary to solve the system:

$$(S_1) : \begin{cases} m^{a,a}x + m^{a,b}y + t(m^{a,a}\alpha + m^{a,b}\beta) &= m^{a,a}x + m^{a,b}y + \lambda t\alpha, \\ m^{b,a}x + m^{b,b}y + t(m^{b,a}\alpha + m^{b,b}\beta) &= m^{b,a}x + m^{b,b}y + \lambda t\beta, \\ \left[z + yt\alpha + \gamma t + \frac{\alpha\beta}{2}t^2 \right] \det(M_\sigma) + P_\sigma(x + t\alpha, y + t\beta) &= \\ z \det(M_\sigma) + \lambda t\alpha(m^{b,a}x + m^{b,b}y) + \gamma\lambda t + \frac{\alpha\beta}{2}(\lambda t)^2 + P_\sigma(x, y). \end{cases}$$

Since the vector (α, β) is an eigenvector associated to the eigenvalue λ , the first two equations are verified. It remains to consider the third equation. It is solved as follows:

$$P_\sigma(x + t\alpha, y + t\beta) = P_\sigma(x, y) + \frac{m^{a,a}m^{b,a}}{2}(2x - 1 + t\alpha)t\alpha + \frac{m^{a,b}m^{b,b}}{2}(2y - 1 + t\beta)t\beta + m^{a,b}m^{b,a}t(\alpha y + \beta x + t\alpha\beta) + n_a^{a,b}t\alpha + n_b^{a,b}t\beta.$$

We must respectively cancel the terms in t^2 , x , y and t in the third line of system (S_1) . Thus, we must solve the system:

$$(S_2) : \begin{cases} \frac{\alpha\beta}{2}\lambda^2 &= \frac{\alpha\beta}{2}\det(M_\sigma) + \alpha^2\frac{m^{a,a}m^{b,a}}{2} + \beta^2\frac{m^{a,b}m^{b,b}}{2} + \alpha\beta m^{a,b}m^{b,a}, \\ \lambda\alpha m^{b,a} &= m^{a,a}m^{b,a}\alpha + m^{a,b}m^{b,a}\beta, \\ \lambda\alpha m^{b,b} &= \alpha\det(M_\sigma) + m^{a,b}m^{b,b}\beta + m^{a,b}m^{b,a}\alpha, \\ \gamma\lambda &= \gamma\det(M_\sigma) - \alpha\frac{m^{a,a}m^{b,a}}{2} - \beta\frac{m^{a,b}m^{b,b}}{2} + n_a^{a,b}\alpha + n_b^{a,b}\beta. \end{cases}$$

Since (α, β) is an eigenvector of the matrix, the first three equations of system (S_2) are always satisfied. Indeed, we observe:

$$\begin{cases} (m^{a,a}\alpha + m^{a,b}\beta)(m^{b,a}\alpha + m^{b,b}\beta) &= (\lambda\alpha) \cdot (\lambda\beta) = \alpha\beta\lambda^2, \\ \text{et } m^{a,a}m^{b,b} - m^{a,b}m^{b,a} &= \det(M_\sigma). \end{cases}$$

The last line of system (S_2) has a solution if $\lambda \neq \det(M_\sigma)$ and we find (1):

$$\gamma = \frac{\alpha}{\lambda - \det(M_\sigma)} \left(n_a^{a,b} - \frac{m^{a,a}m^{b,a}}{2} \right) + \frac{\beta}{\lambda - \det(M_\sigma)} \left(n_b^{a,b} - \frac{m^{b,a}m^{b,b}}{2} \right).$$

□

For example, we can define these flows for the Fibonnaci substitution:

$$\Phi_\phi^t \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + t\frac{1}{\phi} \\ y + t\frac{1}{\phi^2} \\ z + \frac{t(t+1)}{2\phi^3} + \frac{1}{\phi}yt \end{bmatrix} \quad \text{and} \quad \Phi_{-1/\phi}^t \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + t\frac{1}{\phi^2} \\ y - t\frac{1}{\phi} \\ z + \frac{1}{\phi^2}t - \frac{t(t-1)}{2\phi^3} + \frac{1}{\phi^2}yt \end{bmatrix}. \quad (10)$$

3 Example of a special niltranslation

In this section, we consider the left action of the matrix: $\underline{g}_\theta = \begin{pmatrix} 1 & \frac{-1}{\phi} & \theta \\ 0 & 1 & \frac{-1}{\phi^2} \\ 0 & 0 & 1 \end{pmatrix}$ on the group \underline{X} .

We choose a fundamental domain of the quotient space \underline{X} depending on a parameter s :

$$\mathbf{X}^s = \left\{ \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } s \leq x \leq 1+s, \ -s-1 \leq y \leq -s \text{ and } z \in [0, 1] \right\}.$$

Recall that, since the matrix \mathbf{n} introduced in the previous section, is in the center of the group, we are free to quotient by the extremal coordinate “ z ” modulo 1 at any time. In particular, we can consider θ modulo 1.

We restrict ourselves to study the action of \underline{g}_θ on: $\mathbf{Y}^s = \left\{ \mathbf{x} = \begin{bmatrix} -y \\ y \\ z \end{bmatrix} \text{ such that } -s-1 \leq y \leq -s \text{ and } z \in [0, 1] \right\}.$

Since $-\frac{1}{\phi} - \frac{1}{\phi^2} \in \mathbb{Z}$, the element \underline{g}_θ acts by translation on \mathbf{Y}^s . Our goal will be to induce this application on:

$$\mathbf{Y}_{\text{Ind}}^s = \left\{ \mathbf{x} = \begin{bmatrix} -y \\ y \\ z \end{bmatrix} \text{ such that } -s-1 \leq y \leq -s-1 + \frac{1}{\phi^2} \text{ and } z \in [0, 1] \right\}.$$

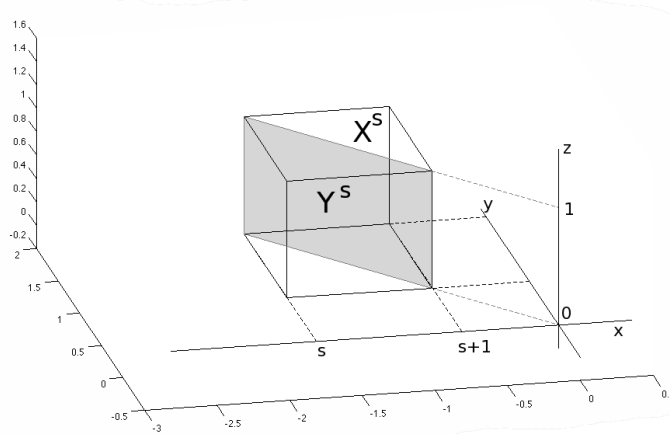


Figure 4: Representation of \mathbf{X}_s and \mathbf{Y}_s .

We will prove the following result at the end of this section:

Proposition 4. *For any parameters $(s, s') \in \mathbb{R}^2$ and any angle $\theta \in \mathbb{R}$, there exists a map, called renormalization, $\tilde{\Phi} : \mathbf{Y}_{\text{Ind}}^s \mapsto \mathbf{Y}^{s'}$ and an angle θ' , such that the first return application of \underline{g}_θ on $\mathbf{Y}_{\text{Ind}}^s$ is conjugated via $\tilde{\Phi}$ to the action of $\underline{g}_{\theta'}$ on $\mathbf{Y}^{s'}$. The angle θ' is given by: $\theta' = \phi^2\theta + \phi^2(s+1) - (s'+1)$.*

In particular, for the parameters $s = s' = -1$ and $\theta = 0$, the action \underline{g}_θ on \mathbf{Y}^{-1} is conjugated to the application $T : (\mathbb{R}/\mathbb{Z})^2 \longrightarrow (\mathbb{R}/\mathbb{Z})^2$, defined by:

$$T(y, z) = (y - \frac{1}{\phi^2} \bmod 1, z + \psi(y) \bmod 1) \text{ where } \psi(y) = -\phi y + \frac{-1}{\phi} \text{ if } 0 \leq y \leq \frac{1}{\phi^2} \text{ and } \psi(y) = -\frac{y}{\phi} \text{ otherwise.}$$

The above calculations assure us that this application is self-induced. A direct calculation shows that:

$$\psi(0) = \psi(1) = \frac{-1}{\phi} \text{ and } \psi^+ \left(\frac{1}{\phi^2} \right) - \psi^- \left(\frac{1}{\phi^2} \right) = -\frac{1}{\phi^3} - \left(-\frac{2}{\phi^2} \right) = -1.$$

The application ψ defines a continuous and Lipschitz map in the torus into itself of degree 1. Thus after the work of H. Furstenberg [10], the system is uniquely ergodic. We also shown that it is self-induced.

We put $p(y) = -\frac{1}{2}y^2 - \frac{1}{2}y$, then $\psi(y) = p(y - \frac{1}{\phi^2} \bmod 1) - p(y) - y + \frac{1}{2\phi^3}$ for all $y \in [0, 1]$. Thus the map T is conjugate with the application of the torus $(\mathbb{R}/\mathbb{Z})^2$ into itself defined by:

$$(y, z) \mapsto \left(y - \frac{1}{\phi^2} \bmod 1, z - y + \frac{1}{2\phi^3} \bmod 1 \right),$$

which is self-induced. We have therefore proved the following result:

Proposition 2. *Let ϕ be the golden mean. The dynamical system given by the application defined from $(\mathbb{R}/\mathbb{Z})^2$ into itself by $(y, z) \mapsto \left(y + \frac{1}{\phi^2}, z + y - \frac{1}{2\phi^3} \right)$, is self-induced, minimal and uniquely ergodic.*

Proof of Proposition 4. Let us write explicitly how the matrix \underline{g}_θ acts on the fundamental domain \mathbf{X}^s .

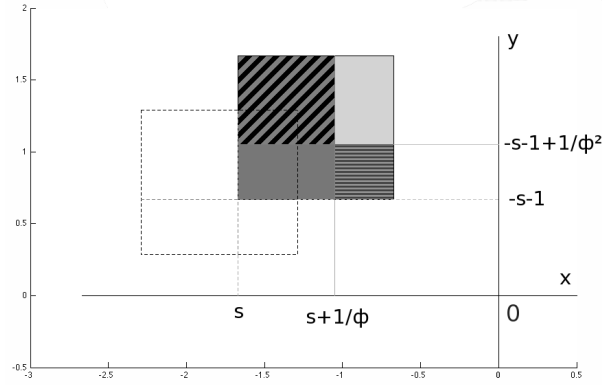


Figure 5: Projection of the four areas of \mathbf{X}^s which act on \underline{g}_θ .

$$\begin{aligned} \text{If } \begin{cases} x \geq s + \frac{1}{\phi} \\ y \geq -s - 1 + \frac{1}{\phi^2} \end{cases} & : \mathbf{x} \mapsto \begin{bmatrix} x - \frac{1}{\phi} \\ y - \frac{1}{\phi^2} \\ z - \frac{y}{\phi} + \theta \end{bmatrix}, \text{ if } \begin{cases} x \geq s + \frac{1}{\phi} \\ y \leq -s - 1 + \frac{1}{\phi^2} \end{cases} & : \mathbf{x} \mapsto \begin{bmatrix} x - \frac{1}{\phi} \\ y - \frac{1}{\phi^2} + 1 \\ z + x - \frac{y}{\phi} + \theta - \frac{1}{\phi} \end{bmatrix}, \\ \text{if } \begin{cases} x \leq s + \frac{1}{\phi} \\ y \geq -s - 1 + \frac{1}{\phi^2} \end{cases} & : \mathbf{x} \mapsto \begin{bmatrix} x - \frac{1}{\phi} + 1 \\ y - \frac{1}{\phi^2} \\ z - \frac{y}{\phi} + \theta \end{bmatrix} \text{ and if } \begin{cases} x \leq s + \frac{1}{\phi} \\ y \leq -s - 1 + \frac{1}{\phi^2} \end{cases} & : \mathbf{x} \mapsto \begin{bmatrix} x - \frac{1}{\phi} + 1 \\ y - \frac{1}{\phi^2} \\ z + x - \frac{y}{\phi} + \theta + \frac{1}{\phi^2} \end{bmatrix}. \end{aligned}$$

We fix parameters (s, s', θ) in \mathbb{R}^3 and we put: $\begin{cases} \mathcal{S} &= \{(y, z) \text{ such that } 0 \leq y \leq 1 \text{ and } 0 \leq z \leq 1\}, \\ \mathcal{S}_{\text{Ind}} &= \{(y, z) \text{ such that } 0 \leq y \leq \frac{1}{\phi^2} \text{ and } z \in [0, 1]\}. \end{cases}$

By “forgetting” for the moment, the first coordinated, the translation by \underline{g}_θ on $\underline{\mathbf{X}}$ is conjugate to an application T_s of $[-s-1, -s] \times [0, 1]$ into itself defined by:

$$\begin{cases} T_s(y, z) &= \left(y - \frac{1}{\phi^2} + 1, z - \phi y + \theta - \frac{1}{\phi} \bmod 1 \right) & \text{if } -s-1 \leq y \leq -s-1 + \frac{1}{\phi^2}, \\ T_s(y, z) &= \left(y - \frac{1}{\phi^2}, z - \frac{y}{\phi} + \theta \bmod 1 \right) & \text{if } -s-1 + \frac{1}{\phi^2} \leq y \leq -s. \end{cases}$$

The application T_s is conjugate by translation, to an application T^s of \mathcal{S} into itself defined by:

$$\begin{cases} T^s(y, z) = \left(y - \frac{1}{\phi^2} + 1, z - \phi y + \theta - \frac{1}{\phi} + (s+1)\phi \bmod 1 \right) & \text{if } 0 \leq y \leq \frac{1}{\phi^2}. \\ T^s(y, z) = \left(y - \frac{1}{\phi^2}, z - \frac{y}{\phi} + \theta + (s+1)/\phi \bmod 1 \right) & \text{if } \frac{1}{\phi^2} \leq y \leq 1. \end{cases}$$

We define the first return of the map T^s of \mathcal{S}_{Ind} into itself as follows:

$$T_{\text{Ind}}^s(y, z) = (T^s)^{n_{y,z}}(y, z) \text{ where } n_{y,z} = \inf \left\{ n \in \mathbb{N}^+; (T^s)^n(y, z) \in \mathcal{S}_{\text{Ind}} \right\}.$$

It is clear that $n_{y,z} = n_y$ only depends on y (neither z , nor s , nor θ), a simple calculation gives us:

$$n_y = 2 \text{ if } 0 \leq y \leq \frac{1}{\phi^4} \text{ and } n_y = 3 \text{ if } \frac{1}{\phi^4} \leq y \leq \frac{1}{\phi^2}.$$

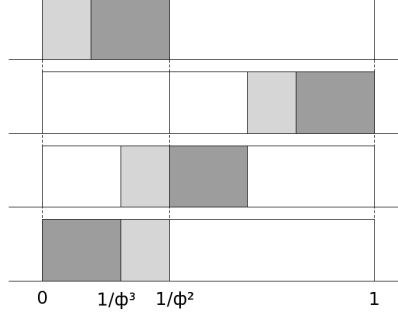


Figure 6: Calculation of n_y .

A direct calculation then yields the expression of T_{Ind}^s :

$$\begin{aligned} T_{\text{Ind}}^s(y, z) &= \left(y + \frac{1}{\phi^3}, z - y \left(\frac{1}{\phi} + \phi \right) + 2\theta + (s+1) \left(\frac{1}{\phi} + \phi \right) \bmod 1 \right) \text{ if } 0 \leq y \leq \frac{1}{\phi^4}, \\ T_{\text{Ind}}^s(y, z) &= \left(y - \frac{1}{\phi^4}, z - y \left(\frac{2}{\phi} + \phi \right) + 3\theta - \frac{1}{\phi^4} + (s+1) \left(\frac{2}{\phi} + \phi \right) \bmod 1 \right) \text{ if } \frac{1}{\phi^4} \leq y \leq \frac{1}{\phi^2}. \end{aligned}$$

We consider the application Φ from \mathcal{S}_{Ind} into \mathcal{S} :

$$\Phi(y, z) = (\phi^2 y, ay^2 + by + z \bmod 1) \text{ and } \Phi^{-1}(y, z) = (\phi^{-2}y, z - \frac{a}{\phi^4}y^2 - \frac{b}{\phi^2}y \bmod 1).$$

The application $\underline{T}^s = \Phi \circ T_{\text{Ind}} \circ \Phi^{-1}$ of \mathcal{S} into itself, obtained by the transfer function Φ is:

If $0 \leq y \leq \frac{1}{\phi^2}$:

$$\begin{aligned} \underline{T}^s(y, z) &= \Phi \circ T_{\text{Ind}}^s(\phi^{-2}y, z - \frac{a}{\phi^4}y^2 - \frac{b}{\phi^2}y \bmod 1) \\ &= \Phi \circ \left(\phi^{-2}y - \frac{1}{\phi^4} + \frac{1}{\phi^2}, z - \frac{a}{\phi^4}y^2 - \frac{b}{\phi^2}y - \phi^{-2}y \left(\frac{1}{\phi} + \phi \right) + 2\theta + (s+1) \left(\frac{1}{\phi} + \phi \right) \bmod 1 \right) \\ &= \left(y - \frac{1}{\phi^2} + 1, z - \frac{a}{\phi^4}y^2 - \frac{b}{\phi^2}y - \phi^{-2}y \left(\frac{1}{\phi} + \phi \right) + 2\theta + (s+1) \left(\frac{1}{\phi} + \phi \right) \right. \\ &\quad \left. + a(\phi^{-2}y - \frac{1}{\phi^4} + \frac{1}{\phi^2})^2 + b(\phi^{-2}y - \frac{1}{\phi^4} + \frac{1}{\phi^2}) \bmod 1 \right) \\ &= \left(y - \frac{1}{\phi^2} + 1, z - \phi^{-2}y \left(\frac{1}{\phi} + \phi - \frac{2a}{\phi^3} \right) + \frac{b}{\phi^3} + \frac{a}{\phi^6} + 2\theta + (s+1) \left(\frac{1}{\phi} + \phi \right) \bmod 1 \right), \end{aligned}$$

and if $\frac{1}{\phi^2} \leq y \leq 1$:

$$\begin{aligned} \underline{T}^s(y, z) &= \Phi \circ T_{\text{Ind}}^s(\phi^{-2}y, z - \frac{a}{\phi^4}y^2 - \frac{b}{\phi^2}y \bmod 1) \\ &= \Phi \circ \left(\phi^{-2}y - \frac{1}{\phi^4}, z - \frac{a}{\phi^4}y^2 - \frac{b}{\phi^2}y - \phi^{-2}y \left(\frac{2}{\phi} + \phi \right) + 3\theta - \frac{1}{\phi^4} + (s+1) \left(\frac{2}{\phi} + \phi \right) \bmod 1 \right) \\ &= \left(y - \frac{1}{\phi^2}, z - \frac{a}{\phi^4}y^2 - \frac{b}{\phi^2}y - \phi^{-2}y \left(\frac{2}{\phi} + \phi \right) + 3\theta - \frac{1}{\phi^4} + (s+1) \left(\frac{2}{\phi} + \phi \right) \right. \\ &\quad \left. + a(\phi^{-2}y - \frac{1}{\phi^4})^2 + b(\phi^{-2}y - \frac{1}{\phi^4}) \bmod 1 \right) \\ &= \left(y - \frac{1}{\phi^2}, z - \phi^{-2}y \left(\frac{2}{\phi} + \phi + \frac{2a}{\phi^3} \right) + 3\theta - \frac{1}{\phi^4} + \frac{a}{\phi^8} - \frac{b}{\phi^4} + (s+1) \left(\frac{2}{\phi} + \phi \right) \bmod 1 \right). \end{aligned}$$

To get the desired result, we must find θ' such that the function \underline{T}^s belongs to the family of initial functions. Therefore, select parameters a and b such that we can find a θ' such that the following two systems admit a solution:

$$(S_1) : \begin{cases} -\phi &= -\phi^{-2} \left(\frac{1}{\phi} + \phi - \frac{2a}{\phi^3} \right), \\ -\frac{1}{\phi} &= -\phi^{-2} \left(\frac{2}{\phi} + \phi + \frac{2a}{\phi^4} \right), \end{cases}$$

and $(S_2) : \begin{cases} \theta' - \frac{1}{\phi} + (s' + 1)\phi &= \frac{b}{\phi^3} + \frac{a}{\phi^6} + 2\theta + (s + 1) \left(\frac{1}{\phi} + \phi \right), \\ \theta' + (s' + 1)/\phi &= 3\theta - \frac{1}{\phi^4} + \frac{a}{\phi^8} - \frac{b}{\phi^4} + (s + 1) \left(\frac{2}{\phi} + \phi \right). \end{cases}$

The first system has a unique solution : $a = -\phi^3$.

The parameters b and s' are related by: $\frac{1}{\phi} - (s' + 1) = \theta - \frac{1}{\phi^2}b + (s + 1)\frac{1}{\phi}$, which fixes the value of parameters:

$$b = \phi^2\theta + \phi(s + 1) + \phi^2(s' + 1) - \phi.$$

So we find: $\theta' = \frac{b}{\phi^3} + \frac{a}{\phi^6} + 2\theta + (s + 1) \left(\frac{1}{\phi} + \phi \right) + \frac{1}{\phi} - (s' + 1)\phi = \phi^2\theta + \phi^2(s + 1) - (s' + 1)$. \square

4 Proof of Theorem 1

We start by fix a nilflow periodic under renormalization and its associate automorphisms \mathcal{L} . By Proposition 1, there exists σ , an automorphism on \mathbb{F}_2 , defined in (6), such that $\mathcal{L} = \mathfrak{S}_\sigma$. The periodic points of the renormalization flow of L. Flaminio and G. Forni, are semi-simple hyperbolic automorphisms which stabilize the discrete Heisenberg group Γ , and preserve the center, up to a change of orientation. So, σ is a **hyperbolic, unimodular** automorphism, that is to say that we impose on M_σ hypothesis (H):

$$M_\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ admits two reals eigenvalues } \lambda \text{ and } \lambda' \text{ such that } |\lambda\lambda'| = 1 \text{ and } |\lambda| > |\lambda'|. \quad (\mathbf{H})$$

Let (α, β) be a nontrivial eigenvector associated to the eigenvalue λ . We can interchange “ a ” with “ a^{-1} ”, or “ b ” with “ b^{-1} ”, so that we can choose $\alpha \geq 0$ and $\beta \geq 0$, such that $\alpha + \beta = 1$. In this way, if v is an infinite word on $\{a, b\}$ such that $\sigma(v) = v$, then α corresponds exactly to the frequency of occurrence of “ a ” in v , and β to the frequency of occurrence of symbol “ b ”.

We fix (α', β') , a nontrivial eigenvector associated to the eigenvalue λ' . Recall that under these conditions, we have: $\alpha'\beta' < 0$, $\lambda \notin \mathbb{Q}$ and $\lambda' \notin \mathbb{Q}$. To simplify the calculations, we impose that $\alpha'^2 + \beta'^2 = 1$.

According to hypothesis (H), the values α' and β' are nonzero. We fix α' strictly negative. We put $\Delta = \alpha\beta' - \alpha'\beta \neq 0$ and we write Φ_λ and $\Phi_{\lambda'}$ the flows obtained by Proposition 3. We write γ and γ' the reals given by equation (1). We note that Δ is negative because $\beta'\Delta = (\beta')^2\alpha + (-\beta')\alpha'\beta > 0$.

The flows Φ_λ and $\Phi_{\lambda'}$ generate a surface $S = \{\Phi_\lambda^t \circ \Phi_{\lambda'}^s(\mathbf{0}); (t, s) \in \mathbb{R}^2\}$ and we write $x_{t,s} = \Phi_\lambda^t \circ \Phi_{\lambda'}^s(\mathbf{0})$.

Let t_a and t_b be the reals defined by $\begin{cases} (t_a\alpha - 1)\beta' &= t_a\beta\alpha', \\ t_b\alpha\beta' &= (t_b\beta - 1)\alpha', \end{cases} \iff \begin{cases} t_a &= \frac{\beta'}{\beta'\alpha - \alpha'\beta} = \frac{\beta'}{\Delta} > 0, \\ t_b &= \frac{-\alpha'}{\beta'\alpha - \alpha'\beta} = \frac{-\alpha'}{\Delta} > 0. \end{cases}$

We put $d_a = \sqrt{(t_a\alpha - 1)^2 + (t_a\beta)^2}$ and $d_b = \sqrt{(t_b\alpha)^2 + (t_b\beta - 1)^2}$. We write $\mathcal{D} = \mathcal{D}_a \cup \mathcal{D}_b$, where:

$\mathcal{D}_a = \{(\alpha's + t\alpha, \beta's + t\beta); -d_a \leq s < 0 \text{ and } 0 \leq t < t_b\}$ and $\mathcal{D}_b = \{(\alpha's + t\alpha, \beta's + t\beta); 0 \leq s < d_b \text{ and } 0 \leq t < t_a\}$.

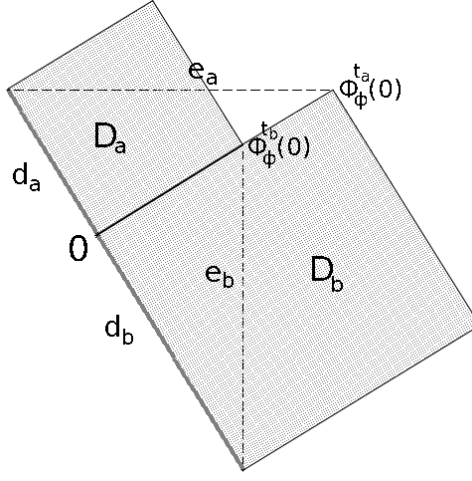


Figure 7: Domain D_τ associated to the substitution of Fibonacci.

Proposition 5. *There exists a polynomial Q_σ of degree 2 in x and y such that $\mathbf{x} = [x, y, z] \in S$ if and only if $z = Q_\sigma(x, y)$.*

By Proposition 3, \mathfrak{S} acts on S by: $\mathfrak{S}(x_{t,s}) = \mathfrak{S} \circ \Phi_\lambda^t \circ \Phi_{\lambda'}^s(\mathbf{0}) = \Phi_\lambda^{\lambda t} \circ \Phi_{\lambda'}^{\lambda' s}(\mathbf{0}) = x_{\lambda t, s \lambda'}$.

We consider the “tile”: $\mathcal{T} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{X}; (x, y) \in \mathcal{D} \text{ and } Q_\sigma(x, y) - 1/2 \leq z < Q_\sigma(x, y) + 1/2 \right\}$.

Proposition 6. *\mathcal{T} is a fundamental domain of X .*

The proof of Proposition 6 is a direct consequence of the fact that D_σ is a fundamental domain for \mathbb{R}^2 ([1]). The aim is to consider the properties of the first return flow in a "good" section. This section will be the surface Σ defined below. Proposition 7 ensures us that this application is self-induced. Then we will see in Proposition 8, that this application is conjugated to a niltranslation, which will have property also to be self-induced. Proposition 9 assures us that this niltranslation is minimal and uniquely ergodic on a surface isomorphic to the torus $(\mathbb{S}^1)^2$. It will complete the proof of Theorem 1.

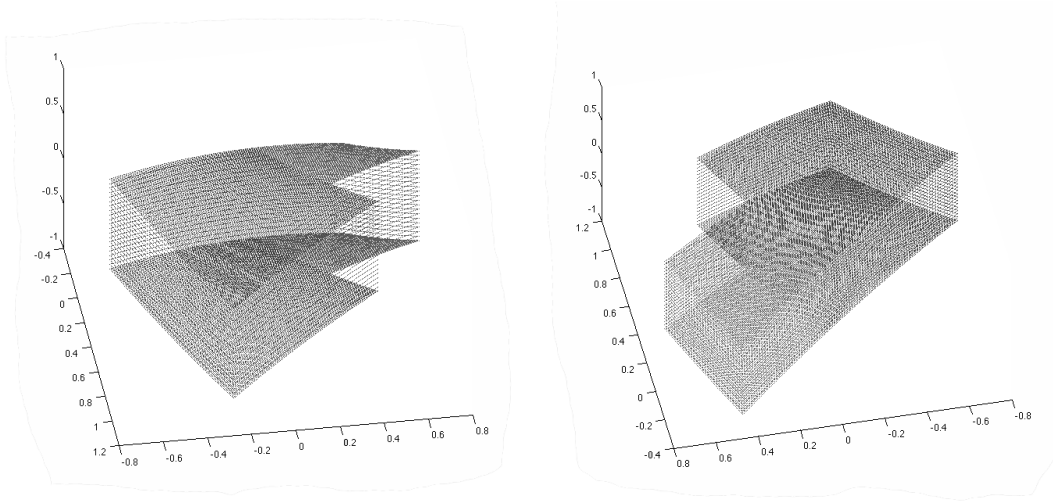


Figure 8: Two angles of the tile associated to the Fibonacci substitution.

Consider the following section: $\Sigma = \left\{ \begin{bmatrix} \alpha' s \\ \beta' s \\ z \end{bmatrix} ; s \in [-d_a, d_b[\text{ and } Q_\sigma(\alpha' s, \beta' s) - \frac{1}{2} \leq z < Q_\sigma(\alpha' s, \beta' s) + \frac{1}{2} \right\}$.

We denote by T_Σ the application of first return of flow Φ_λ from the section Σ into itself.

Proposition 7. *The application T_Σ is self-induced.*

Proof. Let \mathbf{x} be a point of $\mathfrak{S}(\Sigma)$ and $t_{\mathbf{x}} = \inf \{t > 0; \Phi_\lambda^t(\mathbf{x}) \in \mathfrak{S}(\Sigma)\} \in [0, +\infty]$.

We denote by $T_{\mathfrak{S}(\Sigma)}$ the application of $\mathfrak{S}(\Sigma)$ into itself defined by $T_{\mathfrak{S}(\Sigma)}(\mathbf{x}) = \Phi_\lambda^{t_{\mathbf{x}}}(\mathbf{x})$. According to the sign of $\det M_\sigma$, two situations may occur. In all cases, we have $t_{\mathbf{x}} \in \{\min(\lambda t_a, \lambda t_b), \max(\lambda t_a, \lambda t_b)\}$. Assume in the remainder of the proof, that $\det(M_\sigma) > 0$.

Let $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, by Proposition 3:

$$\begin{cases} \text{if } x > 0, & \mathfrak{S}^{-1} \circ T_{\mathfrak{S}(\Sigma)} \circ \mathfrak{S}(\mathbf{x}) = \mathfrak{S}^{-1} \circ \Phi_\lambda^{\lambda t_a} \circ \mathfrak{S}(\mathbf{x}) = \Phi_\lambda^{\frac{1}{\lambda} \lambda t_a}(\mathbf{x}) = \Phi_\lambda^{t_a}(\mathbf{x}) = T_\Sigma(\mathbf{x}), \\ \text{if } x < 0, & \mathfrak{S}^{-1} \circ T_{\mathfrak{S}(\Sigma)} \circ \mathfrak{S}(\mathbf{x}) = \mathfrak{S}^{-1} \circ \Phi_\lambda^{\lambda t_b} \circ \mathfrak{S}(\mathbf{x}) = \Phi_\lambda^{\frac{1}{\lambda} \lambda t_b}(\mathbf{x}) = T_\Sigma(\mathbf{x}). \end{cases}$$

□

We associate to σ , $\mathbf{r}_\sigma = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathfrak{g}$, the matrix $\mathbf{g}_\sigma = \exp \mathbf{r}_\sigma \in \mathbf{X}$ and the niltranslation $T_\sigma : \begin{matrix} \mathbf{X} & \longrightarrow & \mathbf{X} \\ \mathbf{x} & \longmapsto & \mathbf{g}_\sigma \bullet \mathbf{x} \end{matrix}$.

We consider the surfaces: $\mathcal{D} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x + y \in \mathbb{Z} \right\}$ and $D = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x + y = 0 \right\}$.

Proposition 8. *The surface \mathcal{D} immersed in the quotient space, denoted $\underline{\mathcal{D}}$, is a section of the flow Φ_λ with a return time constant, equal to 1. The application of first return flow in this section coincides with the niltranslation \underline{T}_σ on $\underline{\mathcal{D}}$. In addition, the application T_Σ on Σ is measurably conjugate to the niltranslation \underline{T}_σ on $\underline{\mathcal{D}}$ which is also self-induced.*

Proof. The goal is to construct a bijection ψ bi-measurable between the sections $\underline{\mathcal{D}}$ and Σ .

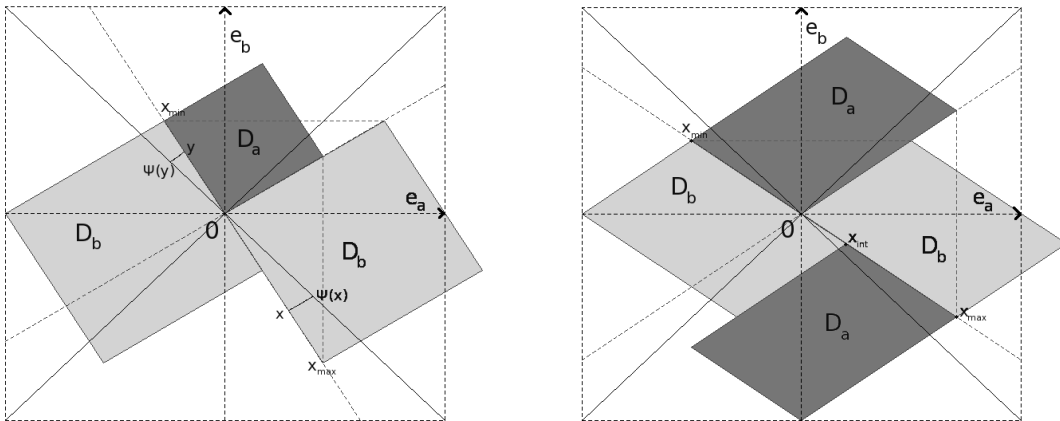


Figure 9: Represation of the tiles in the case where $\alpha' \leq -\beta'$, and $\alpha' > -\beta'$.

We treat only the case where $\alpha' \leq -\beta'$, the other case being analogous. We assume $\alpha' + \beta' < 0$. Let

$$\mathbf{x} = \begin{bmatrix} \alpha' s \\ \beta' s \\ Q_\sigma(\alpha' s, \beta' s) \end{bmatrix} \in \Sigma \text{ with } s \in [-d_a, d_b].$$

We define the time $t^\mathbf{x}$, so that $\Phi_\lambda^{t^\mathbf{x}}(\mathbf{x}) \in D$ satisfies : $\alpha' s + t^\mathbf{x} \alpha = -(\beta' s + t^\mathbf{x} \beta)$, so $t^\mathbf{x} = t^\mathbf{x}(\alpha + \beta) = -(\beta' + \alpha')s$.

The key of the demonstration is to verify that the flow Φ_λ , beginning at a point \mathbf{x} , does not intersect the surface Σ before intersecting the diagonal surface D . For this, *we have just to check the following two conditions:*

$$\text{If } \mathbf{x}_{\max} = \begin{bmatrix} \alpha' d_b \\ \beta' d_b \\ 0 \end{bmatrix}, \text{ then } t^{\mathbf{x}_{\max}} < t_b, \quad (11)$$

$$\text{and if } \mathbf{x}_{\min} = \begin{bmatrix} -\alpha' d_a \\ -\beta' d_a \\ 0 \end{bmatrix}, \text{ then } -t^{\mathbf{x}_{\min}} < t_b. \quad (12)$$

We begin by verifying equation (11).

$$\begin{aligned} t^{\mathbf{x}_{\max}} < t_b &\iff -(\beta' + \alpha')d_b < t_b \iff -\sqrt{(t_b \alpha)^2 + (t_b \beta - 1)^2}(\beta' + \alpha') < t_b \\ &\iff -\sqrt{(\frac{-\alpha'}{\Delta} \alpha)^2 + (\frac{-\alpha'}{\Delta} \beta - 1)^2}(\beta' + \alpha') < \frac{\beta'}{\Delta} \\ &\iff -|\frac{1}{\Delta}| \sqrt{(\alpha' \alpha)^2 + (-\alpha' \beta - \Delta)^2}(\beta' + \alpha') < \frac{\beta'}{\Delta} \\ &\iff \sqrt{(\alpha' \alpha)^2 + (\alpha \beta')^2}(\beta' + \alpha') > \beta' \iff \alpha(\beta' + \alpha') > \beta'. \end{aligned}$$

Thus, it is sufficient to verify that: $\beta' < \alpha \beta' < \alpha \beta' + \alpha \alpha'$ as $\alpha > 0$, $\alpha' > 0$ and $\beta' < 0$.

We now prove that equation (12) is satisfied:

$$\begin{aligned} -t^{\mathbf{x}_{\min}} < t_b &\iff -(\beta' + \alpha')(-d_a) < t_b \iff \sqrt{(t_a \alpha - 1)^2 + (t_a \beta)^2}(\beta' + \alpha') < t_b \\ &\iff \sqrt{(\frac{\beta'}{\Delta} \alpha - 1)^2 + (\frac{\beta'}{\Delta} \beta)^2}(\beta' + \alpha') < \frac{\beta'}{\Delta} \\ &\iff |\frac{1}{\Delta}| \sqrt{(\alpha \beta' - \Delta)^2 + (\beta' \beta)^2}(\beta' + \alpha') < \frac{\beta'}{\Delta} \\ &\iff \sqrt{(\alpha' \beta)^2 + (\beta \beta')^2}(\beta' + \alpha') > \beta' \iff \beta(\beta' + \alpha') > \beta'. \end{aligned}$$

We conclude as before. The case $\alpha' + \beta' > 0$ is treated similarly. We must verify the relations:

$$\text{If } \mathbf{x}_{\min} = \begin{bmatrix} -\alpha' d_a \\ -\beta' d_a \\ 0 \end{bmatrix}, \mathbf{x}_{\text{int}} = \begin{bmatrix} \alpha'(d_b - d_a) \\ \beta'(d_b - d_a) \\ 0 \end{bmatrix} \text{ and } \mathbf{x}_{\max} = \begin{bmatrix} \alpha' d_b \\ \beta' d_b \\ 0 \end{bmatrix},$$

then: $t^{\mathbf{x}_{\min}} < t_a$, $-t^{\mathbf{x}_{\text{int}}} < t_a$ and $-t^{\mathbf{x}_{\max}} < t_b$. The map $\psi : \Sigma \longrightarrow D$, defined by $\psi(\mathbf{x}) = \Phi_\lambda^{t^\mathbf{x}}(\mathbf{x})$ is therefor a bijection between Σ and $\psi(\Sigma)$. By construction, $\underline{T}_\sigma = \psi \circ T_\Sigma \circ \psi^{-1}$. \square

We thus arrive at the following result:

Proposition 9. *The niltranslation \underline{T}_σ on \underline{D} is self-induced, minimal and uniquely ergodic.*

Proof. We have already seen in the previous theorem that this application is self-induced. To prove the result, by Theorem A of E. Lesigne, we just have to check that this map is uniquely ergodic. Since this mapping is continuous, we know that there is at least one invariant measure. Suppose there are two, denoted μ_1 and μ_2 . We choose them distinct and ergodic. The map ψ transports this measure into two different measure μ_1^* and μ_2^* on Σ . We denote by Σ_a (respectively Σ_b), the set of elements \mathbf{x} in Σ such that $\mathbf{p}(\mathbf{x}) \in \mathcal{D}_a$ (respectively $\mathbf{p}(\mathbf{x}) \in \mathcal{D}_b$). We define two singular measures ν_1 and ν_2 on \mathcal{T} as follows. For any continuous function f on \mathcal{T} , we define a function \tilde{f} , measurable on Σ by :

$$\tilde{f}(\mathbf{x}) = \int_0^{t_b} f(\Phi_\lambda^t(\mathbf{x})) dt \text{ if } \mathbf{x} \in \Sigma_a, \text{ and } \tilde{f}(\mathbf{x}) = \int_0^{t_a} f(\Phi_\lambda^t(\mathbf{x})) dt \text{ if } \mathbf{x} \in \Sigma_b.$$

The measures ν_1 and ν_2 are then defined by: $\nu_1(f) = \mu_1^*(\tilde{f})$ and $\nu_2(f) = \mu_2^*(\tilde{f})$.

These measures are invariant by the action of the flow Φ_λ . To conclude, we only have to verify that this flow is uniquely ergodic. Theorem B of L. Auslander, L. Green and F. Hahn assures us that it suffices to show that the ratio $\frac{\alpha}{\beta}$ is irrational. If this ratio is rational, this implies that the eigenvalue $\lambda = m^{a,a} + m^{a,b} \frac{\alpha}{\beta}$ is itself rational, which is absurd since it is a root of an irreducible polynomial of degree 2 in $\mathbb{Z}[X]$. \square

5 More about self-induction

We will now consider to a partial converse of Theorem 1, and we will see that there is an obvious obstruction.

We fix a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, α and β as in Theorem 1. We put $\gamma_0 = -\frac{\alpha AC + \beta BD}{2\lambda - 2\det(M)}$. For any $x \in \mathbb{R}$, we defined

a homeomorphism C_x of \underline{X} by $C_x(\mathbf{y} \bullet \Gamma) = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \bullet \mathbf{y} \bullet \Gamma$.

For any $x \in \mathbb{R}$, the niltranslations by $\mathbf{g} = \begin{bmatrix} \alpha \\ \beta \\ \gamma + \frac{\alpha\beta}{2} \end{bmatrix}$ and $\mathbf{g}(x) = \begin{bmatrix} \alpha \\ \beta \\ \gamma + \frac{\alpha\beta}{2} + \beta x \end{bmatrix}$ are conjugate via C_x . We fix x_0 such that for any $(n, m) \in \mathbb{Z}^2$, $\gamma + \beta x_0 \neq \frac{\alpha}{\lambda - \det(M)} (n - \frac{AC}{2}) + \frac{\beta}{\lambda - \det(M)} (m - \frac{BD}{2})$. From the work developed in Section 4, the niltranslation by $\mathbf{g}(x_0)$ is self-induced, and is the return map of a nilflow not periodic under renormalization.

In the proof of Proposition 8, we explicitly constructed the renormalization map. A serious problem of our work is that unlike the abelian case, this application is not a morphism. However, we believe that there is a partial converse of the theorem, but it is difficult to imagine what kind of renormalizations involved.

We conclude by constructing an example of self-induced niltranslation, for which the areas of induction does not project well on abelianisation.

To simplify the notation, as we have seen in the section, we will be interested by the application T_ϕ , from $\mathbb{R}^2/\mathbb{Z}^2$ into itself, defined by $T_\phi(x, y) = (x + 1/\phi^2, y + x - 1/(2\phi^3))$. We saw in Section 3 that it is self-induced, and that it was equivalent to consider this application, or the associated niltranslation. We consider the quadratic functions p , q and r , defined for all real x by:

$$p(x) = \phi^2 x^2 / 2 - \phi x / 2 - 1/\phi, \quad q(x) = p(x) + \phi^2 x + 3/2 \text{ and } r(x) = p(x) - \phi^2 x + 1 + 1/(2\phi^3).$$

We defined two areas D_1 and D_2 by:

$$\begin{aligned} D_1 &= \left\{ (x, y) ; p(x) < y \leq p(x) + 1 \text{ and } y \leq \min(q(x), r(x) - 1) \right\} \\ \text{and } D_2 &= \left\{ (x, y) ; p(x) < y \leq p(x) + 1 \text{ and } r(x) - 1 < y \leq r(x) \right\}. \end{aligned}$$

We define R , the application from $D = D_1 \cup D_2$ into itself by $R(\mathbf{x}) = T_\phi(\mathbf{x})$, if $\mathbf{x} \in D_1$, and $R(\mathbf{x}) = T_\phi(\mathbf{x}) - (1, 0)$ if $\mathbf{x} \in D_2$. The dynamical system engendered by R is topologically conjugate to that engendered by T_ϕ .

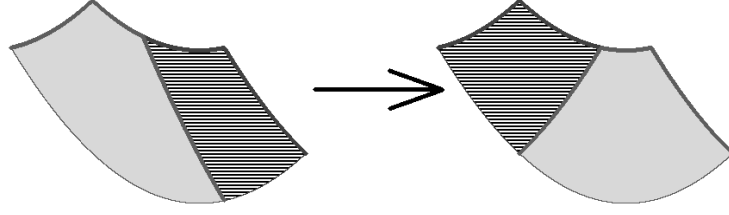


Figure 10: Representation of R .

Proposition 10. *The first return map of R into D_2 , is conjugate to R .*

Proof. We begin by conjugate R with an application defined on an isosceles trapezoid, via $\psi(x, y) = (x, y - p(x))$. We fix

$$\begin{aligned} D'_1 &= \left\{ (x, y) ; 0 < y \leq 1 \text{ and } y \leq \min(\phi^2 x + 3/2, -\phi^2 x + 1/(2\phi^3)) \right\} \\ \text{and } D'_2 &= \left\{ (x, y) ; 0 < y \leq 1 \text{ and } -\phi^2 x + 1/(2\phi^3) < y \leq -\phi^2 x + 1 + 1/(2\phi^3) \right\}. \end{aligned}$$

We define R' , the application from $D' = D'_1 \cup D'_2$ into itself by $R'(x, y) = R'_1(x, y) = (x + \frac{1}{\phi^2}, y)$, if $(x, y) \in D'_1$, and $R'(x, y) = R'_2(x, y) = (x + 1/\phi^2 - 1, y + \phi^2 x - 1/(2\phi^3))$ if $(x, y) \in D'_2$.

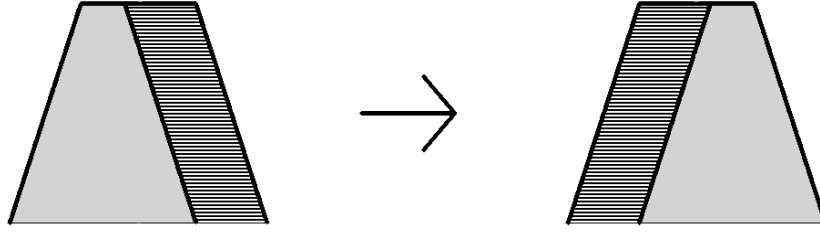


Figure 11: Representation of R' .

The first return map of R into D_2 , is conjugate to the first return map of R' into D'_2 . We can compute that for any \mathbf{x} in D'_2 , it exists an integer $n_{\mathbf{x}} \in \{0, 1, 2, 3\}$ such that the first return map of R' into D'_2 is equal to $R'_1{}^{n_{\mathbf{x}}} \circ R'_2(x, y)$.

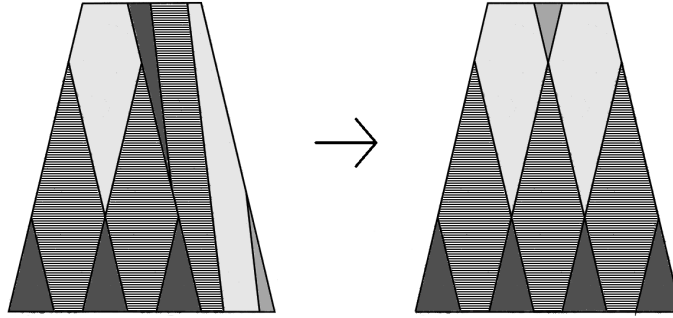


Figure 12: Representation of the first return time map of R' from D'_2 into itself.

We put $\bar{\psi}$ the application from D'_2 into \mathbb{R}^2 defined by $\bar{\psi}(x, y) = (\phi^2 x, y)$. It is not hard to see that $\bar{\psi}(D'_2)$ is a fundamental domain for the torus. To conclude the proof, we have to verify that for any $(x, y) \in \bar{\psi}(D'_2)$, for any

$n \in \{0, 1, 2, 3\}$, we have $\psi \circ R_1'^n \circ R_2' (x/\phi^2, y) = T_\phi(x, y) + (n - 2, 0)$.

$$\begin{aligned} \psi \circ R_1'^n \circ R_2' \begin{pmatrix} x/\phi^2 \\ y \end{pmatrix} &= \psi \circ R_1'^n \begin{pmatrix} x/\phi^2 + 1/\phi^2 - 1 \\ y + x - 1/(2\phi^3) \end{pmatrix} = \psi \begin{pmatrix} x/\phi^2 + (n+1)/\phi^2 - 1 \\ y + x - 1/(2\phi^3) \end{pmatrix} = \begin{pmatrix} x + (n+1) - \phi^2 \\ y + x - 1/(2\phi^3) \end{pmatrix} \\ &= \begin{pmatrix} x + (n+1) - 3 + 1/\phi^2 \\ y + x - 1/(2\phi^3) \end{pmatrix} = T_\phi(x, y) + (n - 2, 0). \end{aligned}$$

These first return applications are very close to those studied by P. Arnoux and C. Mauduit in [4]. \square

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